

**CERTAIN REFLEXIVE SHEAVES ON P_C^n
AND A PROBLEM IN APPROXIMATION THEORY**

BY

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ABSTRACT. This paper establishes a link between certain local problems in the theory of splines and properties of vector bundles and reflexive sheaves on complex projective spaces.

1. Introduction. The purpose of this paper is to establish a link between certain local problems in the theory of splines and properties of vector bundles, or more precisely reflexive sheaves, on complex projective spaces. We shall confine ourselves to splines on \mathbf{R}^n . The algebro-geometric machinery however makes it possible to consider these problems on general real algebraic manifolds, or for that matter real varieties admitting singularities. In this setting a spline function of order μ and degree k is, loosely speaking, a C^μ function on the variety which is piecewise (with respect to some algebraic triangulation) given by polynomials of degree $\leq k$. Thus the variety is subdivided into a finite number of regions whose boundaries are pieces of codimension one subvarieties, and in each region the spline function is given by a polynomial of degree $\leq k$ which agrees to order μ along the boundaries with those on neighboring regions. The basic local problem we wish to consider is, given a fixed triangulation and a point on the variety, to compute the local dimension at the point of the space of spline functions of order μ and degree k . We will therefore restrict our attention to a small neighborhood of the point; essentially working in the ring of germs of C^μ functions at that point.

The author wishes to thank Professor C. Chui for introducing him to these problems.

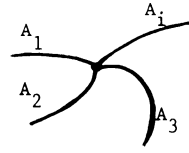
2. The formulation of the problem. We shall begin by focusing on a simple case. Consider in \mathbf{R}^2 a number of polynomial arcs A_1, \dots, A_N emanating from the origin. Each A_i is a piece of the locus $p_i(x, y) = 0$, $p_i(0, 0) = 0$, for some irreducible polynomial $p_i(x, y) \in \mathbf{R}[x, y]$ of degree $d_i \geq 1$. We focus on a small neighborhood about the origin—small enough so that the arcs do not intersect except at the origin and so that they are smooth except perhaps at the origin. Note that we do not assume the p_i are distinct.

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Let $s(x, y)$ denote a spline function of order μ and degree k near the origin for this particular local triangulation. Assuming the arcs are numbered consecutively around the origin, denote by R_i the region between arcs A_i and A_{i+1} for $i = 1, \dots, N$ where $A_{N+1} = A_1$. Our spline function $s(x, y)$ is thus given by a polynomial $f_i(x, y)$ on R_i of degree $\leq k$ with f_i and f_{i+1} agreeing to order μ on A_{i+1} . This means that $f_{i+1} - f_i$ must lie in the $\mu + 1$ st power of the ideal in $\mathbf{R}[x, y]$ defining the irreducible curve of which A_{i+1} is a part. In the case we are dealing with this means

$$f_{i+1} - f_i \in (p_{i+1})^{\mu+1} = (p_{i+1}^{\mu+1}).$$

So

$$f_{i+1} - f_i = q_{i+1} p_{i+1}^{\mu+1}$$

for some polynomial $q_{i+1} \in \mathbf{R}[x, y]$ of degree $\leq k - (\mu + 1)d_i$. If we add these equations

$$(1) \quad \begin{aligned} f_1 - f_N &= q_1 p_1^{\mu+1} \\ f_2 - f_1 &= q_2 p_2^{\mu+1} \\ f_3 - f_2 &= q_3 p_3^{\mu+1} \\ &\vdots \\ f_N - f_{N-1} &= q_N p_N^{\mu+1}, \end{aligned}$$

we get the relation

$$0 = \sum_{i=1}^N q_i p_i^{\mu+1}.$$

This relation was first studied in the context of splines in Wang [6] and later in [1] where it was referred to as the conformality condition.

DEFINITION 2.1. Let $\mathbf{R}P_k$ denote the space of polynomials of degree $\leq k$ in $\mathbf{R}[x, y]$ and let $V_{\mathbf{R}}(p_i, \mu, k) = \{(q_1, \dots, q_N) \text{ with } q_i \in \mathbf{R}P_{k-(\mu+1)d_i} \text{ and } \sum_{i=1}^N q_i p_i^{\mu+1} = 0\}$. Also denote by $S(p_i, \mu, k)$ the space of spline functions at the origin of order μ and degree k with respect to the local triangulation given by the p_i . Note $\dim_{\mathbf{R}} P_k = \binom{k+2}{2}$. \square

PROPOSITION 2.2. $\dim S(p_i, \mu, k) = \binom{k+2}{2} + \dim V_{\mathbf{R}}(p_i, \mu, k)$.

PROOF. Clear. \square

Thus the problem becomes the determination of $\dim V_{\mathbf{R}}(p_i, \mu, k)$ and it is this problem that we reformulate algebro-geometrically.

Without loss of generality we can work over \mathbf{C} and we introduce a third variable z making all polynomials homogeneous. Let P_k denote the set of homogeneous polynomials of degree k in $\mathbf{C}[x, y, z]$. Note $\dim_{\mathbf{C}} P_k = \binom{k+2}{2}$.

For $p_i \in \mathbf{R}[x, y]$, $p_i = \sum_{0 \leq r+l \leq k} a_{r,l} x^r y^l$, we also denote by p_i the associated homogeneous polynomial $\sum_{0 \leq r+l \leq k} a_{r,l} x^r y^l z^{k-r-l}$, and we define

$$V(p_i, \mu, k) = \left\{ (q_1, \dots, q_N), q_i \in P_{k-(\mu+1)d_i} \text{ s.t. } \sum_{i=1}^N q_i p_i^{\mu+1} = 0 \right\}.$$

Clearly $\dim V_{\mathbf{R}}(p_i, \mu, k) = \dim_{\mathbf{C}} V(p_i, \mu, k)$. These changes put us in the setting of complex projective two space $\mathbf{P}_{\mathbf{C}}^2$.

We identify P_k with $H^0(\mathbf{P}_{\mathbf{C}}^2, \mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}(k))$, the space of global sections of the line bundle $\mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}(k)$. Each $p_i^{\mu+1}$ is naturally identified with a global section of $H^0(\mathbf{P}_{\mathbf{C}}^2, \mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}(d_i(\mu+1)))$ or $H^0(\mathcal{O}(d_i(\mu+1)))$ for short. Thus it gives rise to an injective map

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(d_i(\mu+1))$$

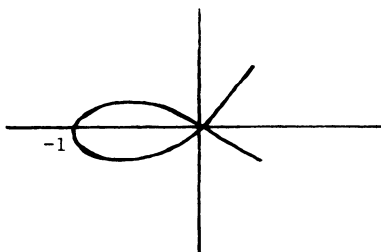
(\mathcal{O} being the trivial line bundle or structure sheaf $\mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}$). We tensor this sequence with $\mathcal{O}(-d_i(\mu+1))$ and consider the direct sum

$$(2) \quad \bigoplus_{i=1}^N \mathcal{O}(-d_i(\mu+1)) \rightarrow \mathcal{O}.$$

The image of this map is a coherent sheaf of ideals \mathcal{G}_X for a closed subscheme $X \in \mathbf{P}_{\mathbf{C}}^2$, which is supported on the intersection of the irreducible curves C_i cut out by the polynomials p_i in $\mathbf{P}_{\mathbf{C}}^2$. This X however may have multiplicities!

Note that \mathcal{G}_X is the coherent sheaf associated to the homogeneous ideal generated by the $p_i^{\mu+1}$ in $\mathbf{C}[x, y, z]$. We can assume that at least two of the p_i are distinct, otherwise the problem is trivial, so that the C_i are not all the same curve and the dimension of X is zero. An example to the contrary would be given by

$$y^2 = x^2(x+1)$$



near the origin where all four arcs are given by the same polynomial.

Now tensor the sequence

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^N \mathcal{O}(-d_i(\mu+1)) \rightarrow \mathcal{G}_X \rightarrow 0,$$

where K is the kernel (a coherent sheaf on $\mathbf{P}_{\mathbf{C}}^2$), with $\mathcal{O}(k)$ to get

$$(3) \quad 0 \rightarrow K(k) \rightarrow \bigoplus_{i=1}^N \mathcal{O}(k - d_i(\mu+1)) \rightarrow \mathcal{G}_X(k) \rightarrow 0,$$

where the notation $K(k)$ means $K \otimes_{\mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}} \mathcal{O}_{\mathbf{P}_{\mathbf{C}}^2}(k)$, etc.

PROPOSITION 2.3. $\dim_{\mathbb{C}} H^0(\mathbf{P}_{\mathbb{C}}^2, K(k)) = \dim_{\mathbb{C}} V(p_i, \mu, k)$.

PROOF. Taking the long exact cohomology sequence of (3) yields

$$0 \rightarrow H^0(K(k)) \rightarrow H^0\left(\bigoplus_{i=1}^N \mathcal{O}(k - d_i(\mu + 1))\right) \xrightarrow{\alpha} H^0(\mathcal{G}_X(k)) \rightarrow \dots$$

We can identify $H^0(\bigoplus_{i=1}^N \mathcal{O}(k - d_i(\mu + 1)))$ with $\bigoplus_{i=1}^N P_{k-d_i(\mu+1)}$ and as $\mathcal{G}_X \subset \mathcal{O}_{\mathbf{P}_{\mathbb{C}}^2}$ we have $H^0(\mathcal{G}_X(k))$ identified with the subspace of P_k consisting of polynomials which “vanish” on X (caution-multiplicities). It is easily seen that α is the map

$$\begin{aligned} \bigoplus_{i=1}^N P_{k-d_i(\mu+1)} &\rightarrow P_k, \\ (q_1, \dots, q_N) &\rightarrow \sum_{i=1}^N q_i p_i^{\mu+1}. \end{aligned}$$

The result follows. \square

The problem now becomes to find $\dim_{\mathbb{C}} H^0(\mathbf{P}_{\mathbb{C}}^2, K(k))$.

THEOREM 2.4. K is locally free and hence a vector bundle of rank $N - 1$ on $\mathbf{P}_{\mathbb{C}}^2$.

PROOF. The sequence $0 \rightarrow K \rightarrow \bigoplus_{i=1}^N \mathcal{O}(-d_i(\mu + 1)) \rightarrow \mathcal{O}_{\mathbf{P}_{\mathbb{C}}^2}$ shows that K is locally a 2nd syzygy sheaf. That is, for a suitable open cover $\{U_{\alpha}\}$ of $\mathbf{P}_{\mathbb{C}}^2$ we have exact sequences

$$0 \rightarrow K|_{U_{\alpha}} \rightarrow \mathcal{O}_{U_{\alpha}}^{\oplus N} \rightarrow \mathcal{O}_{U_{\alpha}}$$

for every α . It follows that the singularity set of K lies in codimension > 2 and is thus empty. Therefore K is locally free on $\mathbf{P}_{\mathbb{C}}^2$. (See Okonek, Schneider, Spindler [4] for details on n th syzygy sheaves and singularity sets.) \square

Thus our problem comes down to finding the global holomorphic sections of a certain vector bundle, $K(k)$, on $\mathbf{P}_{\mathbb{C}}^2$.

3. An asymptotic solution. As first approximation to an answer for our question we consider what happens as $k \rightarrow \infty$, with p_i, μ fixed.

THEOREM 3.1. For k sufficiently large but p_i, μ fixed, we have

$$\dim H^0(\mathbf{P}_{\mathbb{C}}^2, K(k)) = \sum_{i=1}^N \binom{k - d_i(\mu + 1) + 2}{2} - \binom{k + 2}{2} + \dim H^0(\mathcal{O}_X),$$

where we agree $\binom{k - d_i(\mu + 1) + 2}{2} = 0$ if $k - d_i(\mu + 1) < 0$ and where \mathcal{O}_X , the cokernel in $0 \rightarrow \mathcal{G}_X \rightarrow \mathcal{O}_{\mathbf{P}_{\mathbb{C}}^2} \rightarrow \mathcal{O}_X \rightarrow 0$, is the structure sheaf of X and so is zero except at the finite set of points common to the curves C_i .

PROOF. Consider the short exact sequences

$$\begin{aligned} 0 \rightarrow K(k) \rightarrow \bigoplus_{i=1}^N \mathcal{O}(k - d_i(\mu + 1)) \rightarrow \mathcal{G}_X(k) \rightarrow 0, \\ 0 \rightarrow \mathcal{G}_X(k) \rightarrow \mathcal{O}_{\mathbf{P}_{\mathbb{C}}^2}(k) \rightarrow \mathcal{O}_X(k) \rightarrow 0. \end{aligned}$$

Note that $\mathcal{O}_X(k) \cong \mathcal{O}_X$ because the subscheme X is supported at only a finite set of points. The resulting long exact sequences give

$$0 \rightarrow H^0(K(k)) \rightarrow \bigoplus_{i=1}^N H^0(\mathcal{O}(k - d_i(\mu + 1))) \rightarrow H^0(\mathcal{G}_X(k)) \rightarrow H^1(K(k)) \rightarrow 0$$

and

$$0 \rightarrow H^0(\mathcal{G}_X(k)) \rightarrow H^0(\mathcal{O}(k)) \rightarrow H^0(\mathcal{O}_X) \rightarrow H^1(\mathcal{G}_X(k)) \rightarrow 0.$$

By Serre's vanishing theorem $H^1(K(k)) = 0$ and $H^1(\mathcal{G}_X(k)) = 0$ for k sufficiently large. This gives

$$\dim H^0(K(k)) = \sum_{i=1}^N \dim H^0(\mathcal{O}(k - d_i(\mu + 1))) - \dim H^0(\mathcal{G}_X(k))$$

and

$$\dim H^0(\mathcal{G}_X(k)) = \dim H^0(\mathcal{O}_{\mathbf{P}^2}(k)) - \dim H^0(\mathcal{O}_X).$$

Now $\dim H^0(\mathcal{O}(n)) = \binom{n+2}{2}$ for $n \geq 0$ and 0 otherwise. So

$$\dim H^0(K(k)) = \sum_{i=1}^N \left(\binom{k - d_i(\mu + 1) + 2}{2} \right) - \binom{k + 2}{2} + \dim H^0(\mathcal{O}_X). \quad \square$$

COROLLARY 3.2. *The dimension of the space of local spline functions*

$$S(p_i, \mu, k) = \sum_{i=1}^N \left(\binom{k - d_i(\mu + 1) + 2}{2} \right) + \dim H^0(\mathcal{O}_X)$$

for k sufficiently large. \square

We can see that the global geometry intrudes in our calculation. This is because $\dim H^0(\mathcal{O}_X)$ is exactly the intersection of the curves \tilde{C}_i in \mathbf{P}^2 defined by the homogeneous polynomials $p_i(x, y, z)^{\mu+1} = 0$, where the intersections are counted with multiplicities (so the power $\mu + 1$ is properly counted as well). What can we say about $\dim H^0(\mathcal{O}_X)$?

The curves \tilde{C}_i have degree $d_i(\mu + 1)$ so any two distinct ones \tilde{C}_i, \tilde{C}_j intersect in $d_i d_j (\mu + 1)^2$ points (counted properly) by Bezout's theorem. Further intersections can only decrease this number. Thus

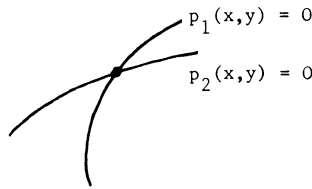
$$1 \leq \dim H^0(\mathcal{O}_X) \leq \underset{\substack{i,j \\ \tilde{C}_i \neq \tilde{C}_j}}{\text{minimum}} d_i d_j (\mu + 1)^2$$

because the curves always have at least the origin in affine x, y -space in common. For example suppose that $N = 2$ and that $d_1 = d_2 = d$ with p_1 and p_2 linearly independent, then for k sufficiently large

$$\begin{aligned} \dim H^0(K(k)) &= 2 \binom{k - d(\mu + 1) + 2}{2} - \binom{k + 2}{2} + d^2(\mu + 1)^2 \\ &= \binom{k - 2d(\mu + 1) + 2}{2} = \dim P_{k-2d(\mu+1)}. \end{aligned}$$

4. Two-dimensional cases. We shall continue to work with spline functions on \mathbf{R}^2 near the origin. Our setting will be in $\mathbf{P}_\mathbb{C}^2$ as above.

We first dispose of the case $N = 2$. (Note that this case is often useful when $N \neq 2$. For example, $N = 4$, where the 4 arcs come from 2 curves meeting at the origin can be reduced to $N = 2$). For $N = 2$ we have that K is a complex line bundle on $\mathbf{P}_\mathbb{C}^2$, but the only such are $\mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(r)$, so $K \cong \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(r)$ for some r .



PROPOSITION 4.1. For $N = 2$ in the two-dimensional case

$$K \cong \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(-(d_1 + d_2)(\mu + 1)).$$

PROOF. Choose a line $L \cong \mathbf{P}_\mathbb{C}^1$ in $\mathbf{P}_\mathbb{C}^2$ that avoids X . This is possible because X has codimension 2. If we restrict the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(-d_1(\mu + 1)) \oplus \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(-d_2(\mu + 1)) \rightarrow \mathcal{G}_X \rightarrow 0$$

to L we get

$$0 \rightarrow K|_L \rightarrow \mathcal{O}_{\mathbf{P}_\mathbb{C}^1}(-d_1(\mu + 1)) \oplus \mathcal{O}_{\mathbf{P}_\mathbb{C}^1}(-d_2(\mu + 1)) \rightarrow \mathcal{O}_{\mathbf{P}_\mathbb{C}^1} \rightarrow 0.$$

Note $\mathcal{G}_X|_L = \mathcal{O}_L \cong \mathcal{O}_{\mathbf{P}_\mathbb{C}^1}$ as $X \cap L = \emptyset$. Computing Chern classes gives

$$c_1(K|_L) = -(d_1 + d_2)(\mu + 1)$$

and the result follows. \square

Note that the sequence

$$0 \rightarrow K \rightarrow \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(-d_1(\mu + 1)) \oplus \mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(-d_2(\mu + 1)) \rightarrow \mathcal{G}_X \rightarrow 0$$

is the Koszul resolution for the complete intersection X .

THEOREM 4.2. For $N = 2$,

$$\begin{aligned} \dim H^0(K(k)) &= \dim H^0(\mathcal{O}_{\mathbf{P}_\mathbb{C}^2}(k - (d_1 + d_2)(\mu + 1))) \\ &= \binom{k - (d_1 + d_2)(\mu + 1) + 2}{2} \end{aligned}$$

if $k \geq (d_1 + d_2)(\mu + 1)$ and 0 otherwise. \square

Thus the example at the end of §3 gives the correct answer for all k not just k sufficiently large.

COROLLARY 4.3. For $N = 2$,

$$\dim_{\mathbf{R}} S(p_i, k, \mu) = \binom{k - (d_1 + d_2)(\mu + 1) + 2}{2} + \binom{k + 2}{2}. \quad \square$$

When $N \geq 3$ things get much more complicated, so we will examine a few special cases before trying to say anything in general.

One interesting case is when $d_i = 1$ for all i which was solved by Chui and Wang in [2]. In this case one can actually work over \mathbf{P}_C^1 since a linear polynomial in $\mathbf{R}[x, y]$ going through the origin is already homogeneous. Moreover, one can assume without loss of generality that the p_i give distinct lines.

Let $p_i = (\alpha_i x + \beta_i y)^{\mu+1}$, $i = 1, \dots, N$ where (α_i, β_i) are pairwise linearly independent. Viewing $p_i \in H^0(\mathbf{P}_C^1, \mathcal{O}_{\mathbf{P}_C^1}(\mu + 1))$, gives rise to a map

$$\mathcal{O}_{\mathbf{P}_C^1}^{\oplus N} \rightarrow \mathcal{O}_{\mathbf{P}_C^1}(\mu + 1)$$

which is onto since the sections p_i have no common zero in \mathbf{P}_C^1 . Twist by $-\mu - 1$ and call the kernel K , this gives the exact sequence

$$(1) \quad 0 \rightarrow K \rightarrow \mathcal{O}_{\mathbf{P}_C^1}^{\oplus N}(-\mu - 1) \rightarrow \mathcal{O}_{\mathbf{P}_C^1} \rightarrow 0$$

where K is an $(N - 1)$ -bundle on \mathbf{P}_C^1 . Let

$$V = \left\{ (q_1 \cdots q_N) \in \mathbf{C}[x, y]^{\oplus N} \text{ with } \deg q_i \leq k - \mu - 1 \text{ s.t. } \sum_{i=1}^N q_i p_i^{\mu+1} \equiv 0 \right\},$$

then it is easy to see that

$$\dim V = \sum_{i=1}^{k-\mu} H^0(\mathbf{P}_C^1, K(\mu + i)).$$

Taking Euler characteristics for the twists of (1) gives

$$\begin{aligned} \chi(K(\mu + i)) &= N\chi(\mathcal{O}_{\mathbf{P}_C^1}(i - 1)) - \chi(\mathcal{O}_{\mathbf{P}_C^1}(\mu + i)) \\ &= Ni - (\mu + i - 1). \end{aligned}$$

Consider the linear maps α_i , $i \geq 1$, in

$$0 \rightarrow H^0(K(\mu + i)) \rightarrow H^0(\mathcal{O}_{\mathbf{P}_C^1}^{\oplus N}(i - 1)) \xrightarrow{\alpha_i} H^0(\mathcal{O}_{\mathbf{P}_C^1}(\mu + i)) \rightarrow H^1(K(\mu + i)) \rightarrow 0.$$

LEMMA 4.4. α_i has maximal rank.

PROOF. We can make a linear change of coordinates so that without loss of generality we can assume all $\alpha_i = 1$ and the β_i are distinct nonzero values. Set

$$A_N^{(\mu+1)} = \begin{bmatrix} 1 & \cdots & 1 \\ \binom{\mu+1}{1}\beta_1 & & \binom{\mu+1}{1}\beta_N \\ \binom{\mu+1}{2}\beta^2 & & \binom{\mu+1}{2}\beta_N^2 \\ \vdots & & \vdots \\ \beta_1^{\mu+1} & & \beta_N^{\mu+1} \end{bmatrix}$$

which is a matrix of size $(\mu + 2) \times N$ and is essentially a Vandermonde determinant so that $A_N^{\mu+1}$ has maximal rank $= \max\{N, \mu + 2\}$.

PROOF. For k sufficiently large the $\dim V$ must be

$$3 \binom{k - \mu + 1}{2} - \binom{k + 2}{2} + \dim H^0(\mathcal{O}_X)$$

by Theorem 3.1 above. Comparing this with the theorem gives the results as $\dim_{\mathbb{C}} H^0(\mathcal{O}_X) = \dim_{\mathbb{C}} \mathcal{O}_{X,o}$. \square

Let us consider another case. We take $N = 3$, $d_i = d = 2$, and $\mu = 0$. We define K in the usual way as the kernel in the sequence

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^2}(-2) \rightarrow \mathcal{G}_X \rightarrow 0$$

where X is the subscheme which is the intersection of the quadrics Q_i given by $p_i(x, y, z) = 0$ in $\mathbb{P}_{\mathbb{C}}^2$. We shall assume that p_1, p_2, p_3 are linearly independent.

K is then a vector bundle of rank 2 on $\mathbb{P}_{\mathbb{C}}^2$ and

PROPOSITION 4.7. K is semistable.

PROOF. We have that $c_1(K) = -6$, so we define K_{norm} to be $K(3)$. It is well known (see [4]) that K is semistable if and only if $H^0(K_{\text{norm}}(-1)) = 0$ and stable if and only if $H^0(K_{\text{norm}}) = 0$.

We consider the sequence

$$0 \rightarrow K(2) \rightarrow \mathcal{O}^{\oplus 3} \rightarrow \mathcal{G}_X(2) \rightarrow 0$$

and the resulting long exact sequence

$$0 \rightarrow H^0(K(2)) \rightarrow \mathbb{C}^3 \rightarrow H^0(\mathcal{G}_X(2)) \\ (c_1, c_2, c_3) \rightarrow \sum c_i p_i.$$

But we have assumed p_i are independent so $0 = H^0(K(2)) = H^0(K_{\text{norm}}(-1))$. \square

Say p_1, p_2, p_3 intersect in $s = 1, 2$, or 3 simple points. Then it is easy to see that

$$c_2(K) = 12 - s$$

and

$$\Delta(K) = c_1^2(K) - 4c_2(K) = -12 + 4s = -8, -4, \text{ or } 0,$$

$$\chi(K) = s - 1 = 0, 1, \text{ or } 2 \quad (\text{use the Riemann-Roch Theorem}).$$

We then compute

$$H^0(K) = 0, \quad H^1(K) = 0, \quad H^2(K) = s - 1,$$

$$H^0(K(1)) = 0, \quad H^1(K(1)) = 3 - s, \quad H^2(K(1)) = 0,$$

$$H^0(K(2)) = 0, \quad H^1(K(2)) = 3 - s, \quad H^2(K(2)) = 0,$$

these last as $H^2(K(r)) = H^1(\mathcal{G}_X(r))$ for $r \geq 1$. Now choose a line $L \cong \mathbb{P}_{\mathbb{C}}^1$ disjoint from X and consider the sequences

$$0 \rightarrow K(r)|_L \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}^{\oplus 3}(r-2) \rightarrow \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^1}(r) \rightarrow 0$$

and

$$0 \rightarrow K(r-1) \rightarrow K(r) \rightarrow K(r)|_L \rightarrow 0.$$

The first sequence shows $K|_L \cong \mathcal{O}_{\mathbf{P}^1}(-a_1) \oplus \mathcal{O}_{\mathbf{P}^1}(-a_2)$ with $a_1 + a_2 = 6$ and $a_i \geq 2$. Thus either $a_1 = a_2 = 3$ or $a_1 = 4$ and $a_2 = 2$. The second sequence shows that for $r \geq 3$ the map $H^1(K(r-1)) \rightarrow H^1(K(r))$ is onto as $H^1(K(r)|_L)$ is then 0.

PROPOSITION 4.8. *For $s = 3$ above, the bundle K is $\cong \mathcal{O}_{\mathbf{P}^2}(-3) \oplus \mathcal{O}_{\mathbf{P}^2}(-3)$.*

PROOF. $H^1(K(k)) = 0$ for all k , which can only occur if K is a sum of line-bundles. \square

On the other hand for $s = 1$ or 2 the bundle K definitely does not split.

To get a complete answer in this case ($N = 3, d_i = d = 2, p_i$ linearly independent, $\mu = 0$) we make use of Noether's "AF + BG" Theorem (see [3]).

THEOREM 4.9. *In the situation above we have for $r \geq 3$,*

$$\dim H^0(K(k)) = \begin{cases} k^2 - 3k + 2 & \text{if } s = 3, \\ k^2 - 3k + 1 & \text{if } s = 2 \\ k^2 - 3k & \text{if } s = 1. \end{cases}$$

PROOF. We are asking for relations $\sum_{i=1}^3 f_i p_i = 0$ where f_i are homogeneous of degree $k - 2$. Given f_3 , when can we find an f_1, f_2 so that $-f_3 p_3 = f_1 p_1 + f_2 p_2$? According to a version of Noether's "AF + BG" theorem this will be possible when the "curve" given by $-f_3 p_3 = 0$ passes through the 4 common points of $p_1 = 0, p_2 = 0$. As p_3 goes through s of them, the requirement is that f_3 pass through the remaining $4 - s$. The number of such f_3 is $\binom{k}{2} - (4 - s)$. The number of ways such a relation can be formed for a fixed f_3 is equal to the number of ways we can write $f_1 p_1 + f_2 p_2 = 0$. But this is the case $N = 2$ solved earlier. The answer was

$$\binom{k - 2}{2}.$$

Thus

$$\begin{aligned} \dim H^0(K(k)) &= \binom{k}{2} - (4 - s) + \binom{k - 2}{2} \\ &= k^2 - 3k - 1 + s. \quad \square \end{aligned}$$

Already in this simple case, the global geometry in $\mathbf{P}_{\mathbb{C}}^2$ intrudes. The dimension of the resulting local spline space is dependent on the number of common solutions in $\mathbf{P}_{\mathbb{C}}^2$ of the polynomials p_i .

Finally, we make some remarks in the general case and give one last example where a complete result can be obtained. We make the following assumptions.

(1) $p_1, \dots, p_N, N \geq 2$, viewed as homogeneous polynomials in $\mathbb{C}[x, y, z]$ are all of degree d and have no common factor so that the subscheme X defined by $\{p_i^{\mu+1}\}_{i=1, \dots, N}$ has dimension 0.

(2)

$$H^0(\mathbf{P}_{\mathbb{C}}^2, \mathcal{G}_X(k)) = \begin{cases} 0, & k < d(\mu + 1), \\ N, & k = d(\mu + 1). \end{cases}$$

This means that no polynomial of degree $< d(\mu + 1)$ "vanishes" on X ("vanishes"

includes multiplicities) and in degree $d(\mu + 1)$ only linear combinations of the p_i “vanish” on X and the p_i are linearly independent.

These conditions are equivalent to the requirement that

$$\bigoplus_{k \geq 0} H^0(\mathcal{G}_X(k))/I_X = (p_1^{\mu+1}, \dots, p_N^{\mu+1})$$

as a graded module over $\mathbf{C}[x, y, z]$ lives only in degree $> d(\mu + 1)$, p_1, \dots, p_N are linearly independent, and $\dim X = 0$. If for example I_X is a saturated ideal (see [5]) then the module above is 0.

As usual we let K be the kernel of the map induced by the p_i ,

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^N \mathcal{O}(-d(\mu + 1)) \rightarrow \mathcal{G}_X \rightarrow 0.$$

PROPOSITION 4.10. *We have*

$$\dim H^0(\mathcal{G}_X(k)) = \begin{cases} 0, & k < d(\mu + 1), \\ N, & k = d(\mu + 1), \end{cases}$$

$$\dim H^1(\mathcal{G}_X(k)) = \begin{cases} \dim H^0(\mathcal{O}_X), & k < 0, \\ \dim H^0(\mathcal{O}_X) - \binom{k+2}{2}, & 0 \leq k < d(\mu + 1), \\ \dim H^0(\mathcal{O}_X) - \binom{k+2}{2} + N, & k = d(\mu + 1), \end{cases}$$

$$\dim H^2(\mathcal{G}_X(k)) = \begin{cases} 0, & k \geq -2, \\ \binom{-k-1}{2}, & k \leq -3, \end{cases}$$

$$\chi(\mathcal{G}_X(k)) = \binom{k+2}{2} - \dim H^0(\mathcal{O}_X).$$

PROOF. This follows from our assumptions and the exact sequence

$$0 \rightarrow \mathcal{G}_X(k) \rightarrow \mathcal{O}_{\mathbf{P}^2}(k) \rightarrow \mathcal{O}_X(k) \rightarrow 0$$

and the fact that $\mathcal{O}_X(k) \cong \mathcal{O}_X$ as $\dim X = 0$. \square

PROPOSITION 4.11. *For all k ,*

$$\chi(K(k)) = N \binom{k - d(\mu + 1) + 2}{2} - \binom{k+2}{2} + \dim H^0(\mathcal{O}_X)$$

and

$$H^0(K(k)) = 0, \quad k \leq d(\mu + 1),$$

$$H^1(K(k)) = 0, \quad k \leq d(\mu + 1),$$

$$H^2(K(k)) = \begin{cases} \chi(K(k)), & k \leq d(\mu + 1), \\ H^1(\mathcal{G}_X(k)), & k \geq d(\mu + 1). \end{cases}$$

PROOF. Use the sequence $0 \rightarrow K(k) \rightarrow \bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}^2}(k - d(\mu + 1)) \rightarrow \mathcal{G}_X(k) \rightarrow 0$ and the preceding proposition. \square

Let H be any hyperplane disjoint from X , i.e. a linear subspace ($\cong \mathbf{P}_C^1$) of \mathbf{P}_C^2 which does not meet X .

PROPOSITION 4.12. *For every k the following restriction sequences are exact*

$$\begin{aligned} 0 \rightarrow K(k)|_H \rightarrow \mathcal{O}_{\mathbf{P}^1}^N(k - d(\mu + 1)) \rightarrow \mathcal{O}_{\mathbf{P}^1}(k) \rightarrow 0, \\ 0 \rightarrow K(k - 1) \rightarrow K(k) \rightarrow K(k)|_H \rightarrow 0. \end{aligned}$$

PROOF. This follows as $H \cap X = \emptyset$. \square

We see that $K|_H$ is a vector bundle on \mathbf{P}_C^1 of rank $N - 1$ and Chern class $-Nd(\mu + 1)$. By Grothendieck's theorem

$$K|_H \cong \bigoplus_{j=1}^{N-1} \mathcal{O}_{\mathbf{P}^1}(-a_j) \quad a_j \in \mathbf{Z}, a_1 \geq a_2 \geq \dots \geq a_{N-1},$$

with $\sum_{j=1}^{N-1} a_j = Nd(\mu + 1)$. From Proposition 4.12 with $k = d(\mu + 1)$ we also see (using the long exact cohomology sequences and Proposition 4.11) that

$$0 = H^0(K(d(\mu + 1))|_H) \cong \bigoplus_{j=1}^{N-1} H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(d(\mu + 1) - a_j)).$$

It follows that $a_j \geq d(\mu + 1) + 1 \forall j$, and so $a_j \leq 2d(\mu + 1) + 2 - N \forall j$. Note that if $N = d(\mu + 1) + 1$ then $a_j = N \forall j$. We can now expand on Proposition 4.11.

PROPOSITION 4.13. $H^2(K(k)) = 0$ for $k \geq 2d(\mu + 1) - N$.

PROOF. For any r we have the long exact sequence

$$\begin{aligned} \dots \rightarrow H^1(K(r - 1)) \rightarrow H^1(K(r)) \xrightarrow{\psi^{(r)}} H^1(K(r)|_H) \\ \xrightarrow{\tau^{(r)}} H^2(K(r - 1)) \rightarrow H^2(K(r)) \rightarrow 0. \end{aligned}$$

If $r \geq 2d(\mu + 1) + 1 - N$ then

$$H^1(K(r)|_H) = \bigoplus_{j=1}^{N-1} H^1(\mathcal{O}_{\mathbf{P}^1}(r - a_j)) = 0$$

because $r - a_j \geq r - (2d(\mu + 1) + 2 - N)$ as $a_j \leq 2d(\mu + 1) + 2 - N$ so $r - a_j \geq -1$. Thus $\tau^{(r)}$ is the zero map for $r \geq 2d(\mu + 1) + 1 - N$ and we have

$$H^2(K(2d(\mu + 1) - N)) \cong H^2(K(2d(\mu + 1) + 1 - N)) \cong \dots \cong 0, \text{ etc.}$$

But by Serre's Vanishing Theorem this must eventually be 0. \square

We now wish to consider the case where

$$\dim H^0(\mathcal{O}_X) = d^2(\mu + 1)^2 - \sum_{j=0}^{N-3} [d(\mu + 1) - 1 - j].$$

For example when $N = 3$, $d = 2$, $\mu = 0$ we are considering 3 linearly independent quadratic polynomials with 3 ($= \dim H^0(\mathcal{O}_X)$) points in common. In general for $N = 3$, $\mu = 0$ we are considering 3 curves of degree d meeting in $d^2 - d + 1$ points.

(Note no curve of degree $< d$ can pass through these points since a curve of degree $d - 1$ meets a curve of degree d in at most $d^2 - d$ points.)

Under these assumptions we have

THEOREM 4.14.

$$K|_H \cong \mathcal{O}_{\mathbf{P}^1}(-2d(\mu + 1) - 2 + N) \oplus \mathcal{O}_{\mathbf{P}^1}(-d(\mu + 1) - 1) \\ \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(-d(\mu + 1) - 1)$$

that is,

$$a_1 = 2d(\mu + 1) + 2 - N \quad \text{and} \quad a_2 = \cdots = a_{N-1} = d(\mu + 1) + 1.$$

PROOF.

$$H^2(K(d(\mu + 1))) = \dim H^0(\mathcal{O}_X) - \binom{d(\mu + 1) + 2}{2} + N$$

while $H^2(K(2d(\mu + 1) - N)) = 0$. From our long exact sequence we have

$$H^1(K(r)|_H) \rightarrow H^2(K(r - 1)) \rightarrow H^2(K(r)) \rightarrow 0.$$

If we start at $H^2(K(2d(\mu + 1) - N))$ we can see that the next lower twist, i.e. $H^2(K(2d(\mu + 1) - 1 - N))$ is bigger by an amount at most

$$\dim H^1(K(2d(\mu + 1) - N)|_H).$$

This space has dimension = 1 for the claimed splitting and dimension = 0 for any other. Working back to $H^2(K(d(\mu + 1)))$ we see that the maximum total increase at any stage is the one given by the claimed splitting. Note that we go back $d(\mu + 1) - N$ steps and that in the last step we have

$$H^1(K(d(\mu + 1) + 1)|_H) \xrightarrow{\tau} H^2(K(d(\mu + 1))) \rightarrow H^2(K(d(\mu + 1) + 1))$$

where for the desired splitting

$$H^1(K(d(\mu + 1) + 1)|_H) \\ = H^1(\mathcal{O}_{\mathbf{P}^1}(-d(\mu + 1) - 1 + N) \oplus H^1(\mathcal{O}_{\mathbf{P}^1}) \oplus \cdots \oplus H^1(\mathcal{O}_{\mathbf{P}^1}))$$

so that the last maximum increase is

$$\dim H^1(\mathcal{O}_{\mathbf{P}^1}(-d(\mu + 1) - 1 + N)) \\ = \dim H^0(\mathcal{O}_{\mathbf{P}^1}(d(\mu + 1) - 1 - N)) = d(\mu + 1) - N$$

using Serre duality on \mathbf{P}^1_C .

Thus the total maximum increase is

$$1 + 2 + \cdots + (d(\mu + 1) - N) = \binom{d(\mu + 1) + 1 - N}{2}$$

and this occurs only for the claimed splitting.

But

$$\begin{aligned} \dim H^2(K(d(\mu + 1))) &= \dim H^0(\mathcal{O}_X) - \binom{d(\mu + 1) + 2}{2} + N \\ &= d^2(\mu + 1)^2 - \sum_{j=0}^{N-3} [d(\mu + 1) - 1 - j] - \binom{d(\mu + 1) + 2}{2} + N \\ &= \binom{d(\mu + 1) + 1 - N}{2}! \end{aligned}$$

Thus the maximum occurs and the only possible splitting is the one claimed. \square

COROLLARY 4.15. *In Proposition 4.13, the map $\tau(r)$ is an injection and $\psi(r) = 0$ for every $r \geq d(\mu + 1) + 1$. \square*

COROLLARY 4.16. $H^1(K(r)) = 0$ all r .

PROOF. We already have this for $r \leq d(\mu + 1)$. But because $\psi(r) = 0 \forall r \geq d(\mu + 1) + 1$ we have

$$\rightarrow H^1(K(r-1)) \rightarrow H^1(K(r)) \xrightarrow{\psi(r)} 0$$

whenever $r \geq d(\mu + 1) + 1$. Thus $H^1(K(d(\mu + 1)))$ maps onto $H^1(K(d(\mu + 1) + 1))$ maps onto $H^1(K(d(\mu + 1) + 2))$, etc.

But $H^1(K(d(\mu + 1))) = 0$. \square

Finally,

$$\chi(K(r)) = \sum_{i=0}^2 (-1)^i \dim_{\mathbb{C}} H^i(\mathbf{P}_{\mathbb{C}}^2, K(r))$$

so

$$\chi(K(r)) = H^0(K(r)) + H^2(K(r))$$

since $H^1 = 0$ by the Corollary. Now $H^2(K(r))$ is known $\forall r$ as a consequence of the Theorem 4.14. Thus we can compute $H^0(K(r))$ all r !

THEOREM 4.18. *In the case considered*

$$H^0(K(r)) = \begin{cases} 0 & \text{for } r \leq d(\mu + 1), \\ (N-2) \binom{r - d(\mu + 1) + 1}{2} & \text{for } d(\mu + 1) + 1 \leq r < 2d(\mu + 1) + 2 - N, \\ (N-2) \binom{r - d(\mu + 1) + 1}{2} + \binom{r - 2d(\mu + 1) + N}{2} & \text{for } r \geq 2d(\mu + 1) + 2 - N. \end{cases}$$

PROOF. We have the exact sequence

$$0 \rightarrow H^0(K(r-1)) \rightarrow H^0(K(r)) \rightarrow H^0(K(r)|_H) \rightarrow 0$$

because H^1 's vanish. For $r = d(\mu + 1) + 1$ we have

$$H^0(K(d(\mu + 1) + 1)) \cong H^0(K(d(\mu + 1) + 1)|_H)$$

and

$$K(d(\mu + 1) + 1)|_H \cong \underbrace{\mathcal{O}_{\mathbf{P}^1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}}_{N-2 \text{ times}} \oplus \mathcal{O}_{\mathbf{P}^1}(N - d(\mu + 1) - 1).$$

Thus $\dim H^0(K(d(\mu + 1) + 1)) = N - 2$. We now continue step by step. \square

Notice that because $H^1(K(r)) = 0 \forall r$ we must have that K is a sum of line bundles on \mathbf{P}_C^2 and in fact

$$K \cong \mathcal{O}_{\mathbf{P}^2}(N - 2d(\mu + 1) - 2) \oplus \mathcal{O}_{\mathbf{P}^2}(-d(\mu + 1) - 1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^2}(-d(\mu + 1) - 1)$$

which agrees with our previous observation in the case $N = 3, \mu = 0, d = 2$.

5. The higher-dimensional case. We now consider spline functions in \mathbf{R}^n . The local problem arises by considering an algebraic triangulation with the origin as a vertex. In each n -simplex having the origin as one vertex we must assign a polynomial of degree $\leq k$ which agrees with those in any adjacent n -simplex to order $\mu + 1$ along their common face which is an $(n - 1)$ -simplex and which is part of the locus $p_i(x_1, \dots, x_n) = 0, p_i(0, \dots, 0) = 0$ for some polynomial in $\mathbf{R}[x_1, \dots, x_n]$. As in the two-dimensional case this means that the difference lies in the $\mu + 1$ st power of the appropriate ideal, $(p_i^{\mu+1}) \subset \mathbf{R}[x_1, \dots, x_n]$ in this case. We then add over all n -simplices with the origin as vertex to get a relation

$$0 = \sum_{i=1}^N q_i p_i^{\mu+1}$$

as before. Let ${}_R P_{k,n}$ be the space of polynomials in n -variables of degree $\leq k$, then

PROPOSITION 5.1.

$$\dim S(p_i, \mu, k) = \binom{k+n}{n} + \dim V_{\mathbf{R}}(p_i, \mu, k)$$

where

$$V_{\mathbf{R}}(p_i, \mu, k) = \left\{ (q_1, \dots, q_N), q_i \in {}_R P_{k-d_i(\mu+1),n} \text{ s.t. } \sum_{i=1}^N q_i p_i^{\mu+1} = 0 \right\}.$$

Note that $\dim {}_R P_{k,n} = \binom{k+n}{n} = \binom{k+n}{k}$. \square

As before we introduce a new variable x_0 to make our polynomials homogeneous, so that we can work in \mathbf{P}_C^n . Let K be the kernel of the map

$$\bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}_C^n}(-d_i(\mu + 1)) \rightarrow \mathcal{O}_{\mathbf{P}_C^n}$$

given by p_1, \dots, p_N viewed as sections of $\mathcal{O}_{\mathbf{P}_C^n}(d_i(\mu + 1))$. We let \mathcal{G}_X be the coherent sheaf of ideals which is the image of this map. X will be a subscheme of \mathbf{P}_C^n supported on the intersection of the hypersurfaces H_i in \mathbf{P}_C^n given by $p_i(x_0, \dots, x_n) = 0$. As in the two-dimensional case

$$\dim_C H^0(\mathbf{P}_C^n, K(k)) = \dim V_{\mathbf{R}}(p_i, \mu, k).$$

THEOREM 5.2. K is a reflexive sheaf on \mathbf{P}_C^n , i.e. $K \cong K^{**}$, the double dual.

PROOF. Recall that a coherent sheaf is reflexive if and only if it is normal and torsion free (see [4]). Clearly K is torsion free because it is a subsheaf of

$$\bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}_C^n}(-d_i(\mu + 1))$$

which is locally free.

The exact sequence

$$0 \rightarrow K \rightarrow \bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}_C^n}(-d_i(\mu + 1)) \rightarrow \mathcal{G}_X \rightarrow 0$$

shows that K is normal, because given any sequence of coherent sheaves

$$0 \rightarrow F \rightarrow E \rightarrow Q \rightarrow 0$$

with E reflexive, then F is normal if Q is torsion free (see [4]). In our case \mathcal{G}_X is clearly torsion free. \square

We could now do the asymptotic results of §3 in this case, but the procedure is identical and we leave it to the reader.

The one case where an explicit answer is possible is when $N \leq n$ and where the codimension of $X \subset \mathbf{P}_C^n$ is N . In other words, when the number of hypersurfaces H_1, \dots, H_N defined by $p_1^{\mu+1}, \dots, p_N^{\mu+1}$ is $\leq n$ and when they intersect in the smallest possible dimension $n - N$. For example $n = 2, N = 2$ which we did in §4 above.

THEOREM 5.3. *Under the assumptions in the preceding paragraph we have*

$$\begin{aligned} \dim_{\mathbf{C}} H^0(K(k)) &= \dim_{\mathbf{R}} V(p_i, \mu, k) \\ &= \sum_{i=2}^N \sum_{1 \leq j_1 < \dots < j_i \leq N} (-1)^i \binom{k + n - (d_{j_1} + \dots + d_{j_i})(\mu + 1)}{n} \end{aligned}$$

where the final term is zero if $k - (d_{j_1} + \dots + d_{j_i})(\mu + 1) < 0$.

PROOF. Let F denote the sheaf $\bigoplus_{i=1}^N \mathcal{O}_{\mathbf{P}_C^n}(-d_i(\mu + 1))$.

Now under our assumptions X is a complete intersection (see Hartshorne [5] or Griffiths [3]). We have the Koszul resolution

$$0 \rightarrow \Lambda^N F \rightarrow \Lambda^{N-1} F \rightarrow \dots \rightarrow \Lambda^2 F \rightarrow \Lambda F \rightarrow F \xrightarrow{\alpha} \mathcal{G}_X \rightarrow 0$$

where the map α is exactly the one we have been considering. It follows that $K(k)$ has a resolution

$$0 \rightarrow \binom{N}{\Lambda F}(k) \rightarrow \binom{N-1}{\Lambda F}(k) \rightarrow \dots \rightarrow \binom{2}{\Lambda F}(k) \rightarrow K(k) \rightarrow 0.$$

Now F is sum of lines bundle and so therefore is $\binom{r}{\Lambda F}(k)$ for all r in fact

$$\binom{r}{\Lambda F}(k) = \bigoplus_{1 \leq j_1 < \dots < j_r \leq N} \mathcal{O}_{\mathbf{P}_C^n}(k - (d_{j_1} + \dots + d_{j_r})(\mu + 1)).$$

Consider the short exact sequence

$$0 \rightarrow K_2 \rightarrow \binom{2}{\Lambda F}(k) \rightarrow K(k) \rightarrow 0$$

where K_2 is the kernel. The long exact cohomology sequence is

$$0 \rightarrow H^0(K_2) \rightarrow H^0\left(\binom{2}{\Lambda F}(k)\right) \rightarrow H^0(K(k)) \rightarrow H^1(K_2) \rightarrow 0.$$

The last 0 is because H^p of a line bundle is zero in \mathbf{P}_C^n for $1 \leq p \leq n - 1$. Thus

$$\dim H^0(K(k)) = \dim H^0\left(\binom{2}{\Lambda F}(k)\right) - \dim H^0(K_2) + \dim H^1(K_2).$$

Now consider

$$0 \rightarrow K_3 \rightarrow \binom{3}{\Lambda F}(k) \rightarrow K_2 \rightarrow 0$$

and we get

$$\begin{aligned} \dim H^0(K_2) - \dim H^1(K_2) &= \dim H^0\left(\binom{3}{\Lambda F}(k)\right) \\ &\quad - \dim H^0(K_3) + \dim H^1(K_3) - \dim H^2(K_3). \end{aligned}$$

This gives

$$\begin{aligned} \dim H^0(K(k)) &= \dim H^0\left(\binom{2}{\Lambda F}(k)\right) - \dim H^0\left(\binom{3}{\Lambda F}(k)\right) \\ &\quad + \dim H^0(K_3) - \dim H^1(K_3) + \dim H^2(K_3). \end{aligned}$$

If we continue in this fashion we get

$$\dim H^0(K(k)) = \sum_{i=2}^{N-1} (-1)^i \dim H^0\left(\binom{i}{\Lambda F}(k)\right) + (-1)^{N-2} \sum_{j=0}^{N-2} \dim H^j(K_{N-1})$$

but $K_{N-1} = \binom{N}{\Lambda F}(k)$ so $H^j(K_{N-1}) = 0$ if $j \neq 0$ in the above sum. It follows that

$$\dim H^0(K(k)) = \sum_{i=2}^N (-1)^i \dim H^0\left(\binom{i}{\Lambda F}(k)\right)$$

and the result is clear. \square

COROLLARY 5.4. *In the case above the dimension of the space of spline functions of order μ and degree k is*

$$\begin{aligned} \dim S(p_i, \mu, k) &= \sum_{i=2}^N \sum_{1 \leq j_1 < \dots < j_i \leq N} (-1)^i \binom{k - (d_{j_1} + \dots + d_{j_i})(\mu + 1) + n}{n} \\ &\quad + \binom{k + n}{n} \end{aligned}$$

where

$$\binom{k - (d_{j_1} + \dots + d_{j_i})(\mu + 1) + n}{n} = 0 \quad \text{if } k < (d_{j_1} + \dots + d_{j_i})(\mu + 1). \quad \square$$

Another way to state our result is

COROLLARY 5.5. *Let $f_i \in \mathbf{R}[x_1, \dots, x_n]$ be polynomials of degree $s_i \geq 1$, $i = 1, \dots, p$. Regarding them as homogeneous of degree s_i in $\mathbf{C}[x_0, \dots, x_n]$ assume that the set of common zeros of the f_i in $\mathbf{P}_{\mathbf{C}}^n$ complex projective n -space has codimension p (e.g. f_1, f_2 so $p = 2$ then f_i can have no common factor).*

Let

$$V(r) = \{(q_1, \dots, q_p) \text{ with } \deg q_i \leq r - s_i \text{ s.t. } \sum q_i f_i \equiv 0\},$$

Then

$$\dim_{\mathbf{R}} V(r) = \sum_{i=1}^p \sum_{\alpha_1 < \dots < \alpha_i} (-1)^i \binom{-s_{\alpha_1} - \dots - s_{\alpha_i} + r + n}{n}$$

where

$$\binom{-s_{\alpha_1} - \dots - s_{\alpha_i} + r + n}{n} = 0 \quad \text{if } -s_{\alpha_1} - \dots - s_{\alpha_i} + r < 0.$$

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